New Algorithms of Neural Fuzzy Relation Systems with Min-implication Composition*

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Abstract. Min-implication fuzzy relation equations based on Boolean-type implications can also be viewed as a way of implementing fuzzy associative memories with perfect recall. In this paper, fuzzy associative memories with perfect recall are constructed, and new on-line learning algorithms adapting the weights of its interconnections are incorporated into this neural network when the solution set of the fuzzy relation equation is non-empty. These weight matrices are actually the least solution matrix and all maximal solution matrices of the fuzzy relation equation, respectively. The complete solution set of min-implication fuzzy relation equation can be determined by the maximal solution set of this equation.

1 Introduction

In recent years, neural networks and fuzzy inference systems have been combined in different ways, and also have got successful applications in the field of pattern recognition systems, fuzzy control and knowledge-based systems [1,3,4,5,6]. The fuzzy associative memory network with max-min composition is first studied by Kosko [3]. Pedrycz has used neural networks for the resolution of max-min fuzzy relation equations (FREs) [1]. The neural network proposed in [1] converges with the aid of a learning algorithm to the maximum solution of sup $-t$ FRE [5], when the solution set of the FRE is non-empty. Because fuzzy inference systems with min-implication composition (where the implication is any Boolean-type implication, which mainly contains R-implication, S-implication and Ql-implication) have good logic foundation [11], and approximate capability [9,10], and it is also shown that in most practical cases these FREs do have non-empty solution sets

* This work is supported by 973 program of China (Project No.2002CB312200), National Science Foundation of China (Project No.60474045), Zhejiang University Scientific Research Foundation for the Returned Overseas Scholars, Zhejiang Province Foundation for the Returned Overseas Scholars, and the SRF for ROCS, SEM.

L. Wang, K. Chen, and Y.S. Ong (Eds.): ICNC 2005, LNCS 3612, pp. 1132–1141, 2005.

[7,8]. Min-implication FREs can also be viewed as a way of implementing fuzzy associative memories (FAM) with perfect recall. So, neural fuzzy relation systems with min-implication composition are better applied into fuzzy modeling than those with $\sup -t$ composition.

In this paper, we extend the neural network for implementing fuzzy relation systems based on $\sup -t$ composition in [5] to min-implication composition. We suppose that the respective constructed FRE has a solution, and provide learning algorithms in order to find the least solution and all maximal solutions of the equation.

The structure of the rest of paper is organized as follows. In section 2, we give a brief review of Boolean-type implications, min-implication FREs and some solvable criteria for min-implication FREs based on Boolean-type implications. A type of neuron that implements min-implication composition FREs and new learning algorithms for solving the least solution and all maximal solutions of FREs are proposed in section 3. In section 4, some simulation results are presented and finally the conclusions are given in section 5.

2 Preliminaries

In this paper we assume that the universe of discourse is a finite set. Suppose that $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_m\}$ and $N_n = \{1, 2, \cdots, n\}, N = \{0, 1, 2, \cdots, n, \cdots\}$. Let $F(X)$ denote the set of all of fuzzy sets in X, $A \in F(X), B \in F(Y), R \in F(X \times Y)$.

Because X and $X \times Y$ are finite, any element of $F(X)$ and $F(X \times Y)$ can be denoted by a vector and a $n \times m$ matrix, respectively.

For a min-implication FRE:

$$A \circ^{\theta} R = B. \tag{1}$$

where A, R and B are $n \times m$ matrix, $m \times k$ matrix and $n \times k$ matrix, respectively. Eq.(1) can be decomposed as a set of k simpler min-implication FREs:

$$A \circ^{\theta} \mathbf{r} = \mathbf{b}. \tag{2}$$

where $r_{m\times 1}$ and $b_{n\times 1}$ are the column vectors of R and B. Let $\zeta(A, \mathbf{b})$ be the solution set of the FRE of the form (2), i.e. $\zeta(A, \mathbf{b}) = \{\mathbf{r} : A \circ^{\theta} \mathbf{r} = \mathbf{b}\}$.

Let $a \hat{\otimes}^{\theta} b$ and $a \breve{\otimes}^{\theta} b$ denote the maximal and the minimal, respectively, solution of the equation $a\theta x = b$ (if they exist).

Based on the above notations, the maximal solution operator (max-SO) $\hat{w}_{\theta}$ and the minimal solution operator (min-SO) $\breve{w}_{\theta}$ are defined as follows:

$$\hat{w}_{\theta}(a, b) = \begin{cases} a \hat{\otimes}^{\theta} b, & a\theta 0 \le b \\ 1, & \text{otherwise}, \end{cases} \tag{3}$$

$$\breve{w}_{\theta}(a, b) = \begin{cases} a \breve{\otimes}^{\theta} b, & a\theta 0 < b \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

For the convenience of denoting the maximal solution matrix, we define the following operator:

$$w_{max}(a,b) = \begin{cases} 1, a > b \\ b, \text{otherwise.} \end{cases} \quad (5)$$

Definition 1. *Let θ be a Boolean-type implication. θ is called a nice Boolean-type implication (NBoolean-type implication) if θ is continuous with respect to its second variable, i.e. $\forall a \in [0,1]$, the function $f(x) = a\theta x : [0,1] \to [0,1]$ is continuous.*

Let θ be an NBoolean-type implication. The mean-SM ($\bar{\Gamma}$), the min-SM ($\breve{\Gamma}$) and the max-SM ($\hat{\Gamma}$) of Eq.(2) are defined in [7,8]. We extend the solvable criteria for any S-implication [7] and R-implication [8] to any Boolean-type implication and present them in the following propositions.

Proposition 1. *Let $A \circ^{\theta} \boldsymbol{r} = \boldsymbol{b}$ be a min-implication FRE of the form (2). If $\forall i \in N_n$, $\exists j \in N_m$, such that $b_i \in I(a_{ij})$, and $b_k \notin I(a_{kj})$, $\forall k \in N_n - \{i\}$, then $\zeta(A, \boldsymbol{b}) \neq \emptyset$.*

Proposition 2. *Let θ be an NBoolean-type implication, $A \circ^{\theta} \boldsymbol{r} = \boldsymbol{b}$ be a min-implication FRE of the form (2). Then, $\zeta(A, \boldsymbol{b}) \neq \emptyset$ iff (if and only if) $A \circ^{\theta} \inf \hat{\Gamma} = \boldsymbol{b}$.*

Proposition 3. *Let θ be an NBoolean-type implication, $A \circ^{\theta} \boldsymbol{r} = \boldsymbol{b}$ be a min-implication FRE of the form (2), $\boldsymbol{t} = sup\breve{\Gamma}$. Then, $\zeta(A, \boldsymbol{b}) \neq \emptyset$ iff $A \circ^{\theta} \boldsymbol{t} \leq \boldsymbol{b}$, and $\boldsymbol{t}$ is the least solution.*

Proposition 4. *Let θ be an NBoolean-type implication, $A \circ^{\theta} \boldsymbol{r} = \boldsymbol{b}$ be a min-implication FRE of the form (2). The following statements are equivalent:*
(i) $\zeta(A, \boldsymbol{b}) = \emptyset$;
(ii) $\exists j \in N_n$ such that $\hat{\Gamma}_{\cdot j} = 1$ and $b_j \neq 1$.

In the following, we propose new learning algorithms to obtain the least solution and all maximal solutions of the FRE of the form (1), respectively.

3 Min-implication Operator Networks and New Algorithms

3.1 A Type of Neuron That Implements Min-implication FREs

Let $D = \{(\mathbf{a}_j, \mathbf{b}_j), j \in N_n\}$ be a set of input-output data. Compositional fuzzy associative memories can be generalized to the finding of the fuzzy relation matrix R for which the following equation holds:

$$\mathbf{a}_j \circ^{\theta} R = \mathbf{b}_j, \quad j = 1, 2, \ldots, n \quad (6)$$

where θ is an NBoolean-type implication.

However, the matrix satisfying (6) doesn't always exist in most case. Even if it exists, the weight matrix satisfying (6) doesn't have to be a solution of the equation. The choice of training sets plays an important role in the generation of their connection weights. But, it is difficult to give ideal training sets D. Therefore, an effective method is to adjust the weights to matching the input-output pairs.

A type of compositional neuron that implements the union-intersection composition of fuzzy relation is proposed by G. B. Stamou and S. G. Tzafestas [5]. In the following, we propose a similar type of neuron for the min-implication composition.

The general structure of a conventional neuron can be shown in Fig.1. The equation that describes this kind of neuron is as follows:

$$y = g(u) = g(f(x)) = g(\sum_{j=1}^{n} w_i x_i + \theta), \tag{7}$$

where θ is a threshold and w_i $(i = 1, 2, \ldots, n)$ are weights that can change on-line with the aid of a learning process.

The compositional neural has the same structure with the neuron of (7) (Fig.1), but it can be described by the equation:

$$y = g(S_{j \in N_n} t(x_i, w_i)), \tag{8}$$

where S is a fuzzy union operator (an s-norm), t is a fuzzy intersection operator (a t-norm) and g is the activation function:

$$g(x) = \begin{cases} 0, \ x \in (-\infty, 0) \\ x, \ x \in [0, 1] \\ 1, \ x \in (1, +\infty). \end{cases} \tag{9}$$

The structure of this neural network is shown in Fig.2.

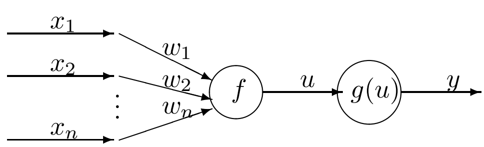

Fig. 1. The structure of the neuron

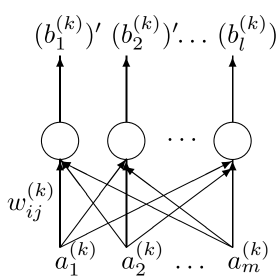

Fig. 2. The structure of the compositional neuron

The equality index proposed in [1] is defined by:
$[(b_i^{(k)})' \equiv b_i^{(k)}] = \frac{1}{2}[((b_i^{(k)})' \to b_i^{(k)}) \wedge (b_i^{(k)} \to (b_i^{(k)})') + ((\overline{b}_i^{(k)})' \to \overline{b}_i^{(k)}) \wedge (\overline{b}_i^{(k)} \to (\overline{b}_i^{(k)})')]$, where "$\to$" denotes the Łukasiwicz implication, the complement (negation) is linear, i.e. $\overline{b}_i^{(k)} = 1 - b_i^{(k)}$. Finally, we obtain the following expression for the equality index:

$$[(b_i^{(k)})' \equiv b_i^{(k)}] = \begin{cases} 1 + (b_i^{(k)})' - b_i^{(k)}, & (b_i^{(k)})' < b_i^{(k)} \\ 1 + b_i^{(k)} - (b_i^{(k)})', & (b_i^{(k)})' > b_i^{(k)} \\ 1, & (b_i^{(k)})' = b_i^{(k)}. \end{cases} \quad (10)$$

The error is defined as follows:

$$E_i^{(k)} = 1 - [(b_i^{(k)})' \equiv b_i^{(k)}].$$

Our goal in learning processes is minimize the total error

$$E^{(k)} = \sum_i E_i^{(k)}. \quad (11)$$

Due to the decomposition of the FRE [7,8], the proposed algorithm converges independently for each neuron. For simplicity and without loss of generality, we only consider the single neuron case. The response of the neuron $(b^{(k)})'$ is given by

$$(b^{(k)})' = \inf_{i \in N_m} (a_i^{(k)} \theta w_i^{(k)}).$$

where $w_i^{(k)}$ $(i \in N_m)$ are the weights of the neuron and the input $a_i^{(k)}$. In this case, the desired output is $b_i^{(k)}$.

3.2 The Learning Algorithm for Solving the Least Solution of the FRE

Based on the neuron network described by the Fig.2, we give the learning algorithm for solving the least solution of the FRE of the form (2) in the following:
Algorithm 1. Step 1. Initialize the weights as $\breve{w}_i^{(1)}(0) = 0$, $i \in N_m$, and give an error constant ϵ and the upper limit value of iteration μ.

Step 2. Input a pair of pattern $(\mathbf{a}_k, \mathbf{b}_k)$, $k \in N_n$, and let $v = 0$.

Step 3. The network on-line weights $\breve{w}_i^{(k)}(i = 1, 2, \ldots, m)$ are adjusted by the following algorithm

$$\breve{w}_i^{(k)}(v+1) = \breve{w}_i^{(k)}(v) + \Delta\breve{w}_i^{(k)}(v), \quad (12)$$

$$\Delta\breve{w}_i^{(k)}(v) = \alpha l_s, \quad (13)$$

$$l_s = \begin{cases} \alpha_1(\breve{w}_\theta(a_{ki}, b_k) - \breve{w}_i^{(k)}(v)), & \breve{w}_i^{(k)}(v) < \breve{w}_\theta(a_{ki}, b_k) \\ \alpha_2(\breve{w}_i^{(k)}(v) - \hat{w}_\theta(a_{ki}, b_k)), & \breve{w}_i^{(k)}(v) > \hat{w}_\theta(a_{ki}, b_k) \\ 0, & \text{otherwise}, \end{cases} \quad (14)$$

where $\breve{w}_\theta$ and $\hat{w}_\theta$ are given by (3) and (4), $0 < \alpha \leq 1$ and $0 \leq \alpha_1, \alpha_2 \leq 1$ are learning rates.

Step 4. Compute the total error $E^{(k)}$.

Step 5. Return to Step 3 until $E^{(k)} < \epsilon$ or $v > \mu$, in this case, the last weights are denoted by $\breve{w}_j^{(k)}, j = 1, 2, \ldots, m$.

Step 6. If $k < n$, return to Step 2 with $k = k + 1$ and $\breve{w}_j^{(k+1)}(0) = \breve{w}_j^{(k)}, j = 1, 2, \ldots, m$.

With this algorithm we obtain the following theorem.

Theorem 1. *Let $D = \{(\boldsymbol{a}_j, \boldsymbol{b}_j), j \in N_n\}$ be a training set, where $\boldsymbol{a}_j$ and $\boldsymbol{b}_j$ are the jth column of the matrix A and B, respectively. The neuron network of Fig.2 with the learning algorithm described by (12),(13) and (14), converges (with $\epsilon = 0$) to the least solution of the FRE:*

$$A \circ^\theta R = B$$

if $\zeta(A, B) \neq \emptyset$ and $\alpha = 1, \alpha_1 = 1, \alpha_2 = 0$.

Proof (of theorem). Without loss of generality, we will prove the Theorem only for one neuron. Since $\zeta(A, B) \neq \emptyset$, it is clear that $\zeta(A, \mathbf{b}) \neq \emptyset$. Then, from Proposition 4 we have that

$\hat{\Gamma}_{\cdot j} = 1$ implies $b_j = 1, \forall j \in N_n$.

From the definition of $\hat{\Gamma}$ we have that

$$\hat{\Gamma}_{ij} = w_{max}(\sup_{k \in N_n} \breve{\Gamma}_{ik}, \bar{\Gamma}_{ij}), \forall i \in N_m, \forall j \in N_n. \tag{15}$$

From (15), it is true that for any $j \in N_n$ there exists $i \in N_m$, such that

$$\hat{\Gamma}_{ij} = \bar{\Gamma}_{ij} \geq \sup_{k \in N_n} \breve{\Gamma}_{ik}.$$

Hence, from the definition of the solution matrices, we have

$$\hat{w}_\theta(a_{ji}, b_j) \geq \sup_{k \in N_n} \breve{w}_\theta(a_{ki}, b_k). \tag{16}$$

From (16) it is true that $\forall j \in N_n, \exists i \in N_m$ such that $\forall v \in N$,

$$\hat{w}_\theta(a_{ji}, b_j) \geq \breve{w}_i^{(j)}(v).$$

From (12)-(14), and $\alpha = \alpha_1 = 1, \alpha_2 = 0$, we have $\forall j \in N_n, \exists i \in N_m$ such that $\forall v \in N$,

$$\breve{w}_i^{(j)}(v+1) = \begin{cases} \breve{w}_\theta(a_{ji}, b_j), \ \breve{w}_\theta(a_{ji}, b_j) \geq \breve{w}_i^{(j)}(v) \\ \breve{w}_i^{(j)}(v), \quad \text{otherwise}, \end{cases}$$

and $\{\breve{w}_i^{(j)}(v)\}(v \in N)$ is a upper bounded and monotonic increasing sequence. Thus, there exists $v_0 \in N$ such that $\forall v > v_0$, it is

$$\breve{w}_i^{(j)}(v) = \sup_{k \in N_n} \breve{w}_\theta(a_{ki}, b_k), i = 1, 2, \ldots, m.$$

Therefore, there exists $v_0 \in N$ such that $\forall v > v_0$, it is true that

$\mathbf{a}_j \circ^\theta \breve{\mathbf{w}}^{(j)}(v) = b_j, \quad \forall j \in N_n.$ □

3.3 The Learning Algorithm for Solving All Maximal Solutions of the FRE

From Algorithm 1 and Theorem 1, we can get the least solution $\mathbf{t} = \sup \breve{\Gamma}$ of the FRE of the form (2). In the following, we give a learning algorithm for calculating all maximal solutions of the FRE. The structure of the single-layer neural network with compositional neurons is also illustrated in Fig.2. For simplicity and without loss of generality, we only consider the single neuron case.

The adaptation of the neural network is supported by the following algorithm.

Algorithm 2. Step 1. Initialize the weights as $\hat{w}_j^{(1)}(0) = 1$, $j \in N_m$, and give an error constant ϵ and the upper limit value of iteration μ.

Step 2. Input $(\mathbf{a}_k, b_k), k \in N_n$, then calculate $N(k)$, where $N(k) = \{j \in N_m : a_{kj}\theta t_j = b_k\}, t_j = \sup_i \breve{w}_\theta(a_{ij}, b_i)$, and let $v = 0$.

Step 3. The weights $\hat{w}_j^{(k)}$, $j = 1, 2, \ldots, m$, are adjusted as follows

$$\hat{w}_j^{(k)}(v+1) = \hat{w}_j^{(k)}(v) - \Delta\hat{w}_j^{(k)}(v), \tag{17}$$

$$\Delta\hat{w}_j^{(k)}(v) = \eta l_s, \tag{18}$$

$$l_s = \begin{cases} \hat{w}_j^{(k)}(v) - (a_{kj}\hat{\otimes} b_k \wedge \hat{w}_j^{(k)}(v)), \; j \in N(k) \\ 0, \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise}, \end{cases} \tag{19}$$

$j = 1, 2, \ldots, m$.

Step 4. Compute the total error $E^{(k)}$.

Step 5. Return to Step 3 until $E^{(k)} < \epsilon$ or $v > \mu$, in this case, the last weights are denoted by $\hat{w}_j^{(k)}, j = 1, 2, \ldots, m$.

Step 6. If $k < n$, return to Step 2 with $k = k+1$ and $\hat{w}_j^{(k+1)}(0)) = \hat{w}_j^{(k)}, j = 1, 2, \ldots, m$.

Step 7. Calculate the set $E_t = \{f \in N_m : f \cap N(1) \cap N(2) \cap \cdots \cap N(n)$ contains at least one element$\}$. Let F be the subset of E_t including all the minimal elements of E_t. For any $f \in F$, adjust the elements of the weight vector $\hat{\mathbf{w}} = \{\hat{w}_1^{(n)}, \hat{w}_2^{(n)}, \ldots, \hat{w}_m^{(n)}\}$ as follows

$$\hat{w}_j^{(f)} = \begin{cases} \hat{w}_j^{(n)}, \; j \in f \\ 1, \qquad \text{otherwise}, \end{cases} \tag{20}$$

$j = 1, 2, \ldots, m$.

With the above algorithm we get the following theorem.

Theorem 2. *Let $D = \{(\boldsymbol{a}_j, b_j), j \in N_n\}$ be a training set, where $\boldsymbol{a}_j$ and b_j are the jth column of the matrix A and the vector $\boldsymbol{b}$, respectively. All the weight vectors $\hat{\boldsymbol{w}}^{(f)} = (\hat{w}_1^{(f)}, \hat{w}_2^{(f)}, \ldots, \hat{w}_m^{(f)})$, $\forall f \in F$, from Algorithm 2, converge (with $\epsilon = 0$) to all maximal solutions of the FRE:*

$$A \circ^\theta \boldsymbol{r} = \boldsymbol{b}$$

if $\zeta(A, \boldsymbol{b}) \neq \emptyset$ and $\eta = 1$.

Proof (of theorem). It is obvious that E_t is a finite poset under the subset inclusion from the definition of E_t. Then F is also finite and for any $f \in E_t$, there is $g \in F$ such that $g \subseteq f$ and the elements of F are not comparable with each other (in other words, the elements of F do not contain each other). In the following, we verify that $M = \{\hat{\mathbf{w}}^{(f)} : \forall f \in F\}$ satisfies the conditions of the theorem.

At first, we prove that $\hat{\mathbf{w}}^{(f)}$, $\forall f \in F$ all are solutions of the FRE of the form (2). If $b_i = 1$, then for any $j \in N(i), t_j = 1$. In this case, $a_{ij} = 0$ and $\hat{w}_j^{(f)} = 1$, and hence $\inf_j(a_{ij}\theta\hat{w}_j^{(f)}) = 1 = b_i$. If $b_i < 1$, then there exists $j_0 \in N_m$ such that $a_{ij_0}\theta t_{j_0} = b_i$, then $j_0 \in N(i)$. In this case, $\inf_j(a_{ij}\theta\hat{w}_j^{(f)}) \leq a_{ij_0}\theta\hat{w}_{j_0}^{(f)} = a_{ij_0}\theta\hat{w}_{j_0}^{(n)} = a_{ij_0}\theta(\inf\{a_{ij_0}\hat{\otimes}^{\theta} b_i : j_0 \in N(i)\}) \leq a_{ij_0}\theta(a_{ij_0}\hat{\otimes}^{\theta} b_i) = b_i$. On the other hand, since $j \in N(i)$, that is $a_{ij}\theta t_j = b_i$, then $a_{ij}\hat{\otimes}^{\theta} b_i \geq t_j$. Thus, $\hat{w}_j^{(f)} = \hat{w}_j^{(n)} = \inf\{a_{ij}\hat{\otimes}^{\theta} b_i : j \in N(i)\} \geq \inf\{t_j : j \in N(i)\} = t_j$. Then, $\inf_j(a_{ij}\theta\hat{w}_j^{(f)}) \geq \inf_j(a_{ij}\theta t_j) = b_i$. So, $\hat{\mathbf{w}}^{(f)} = (\hat{w}_1^{(f)}, \hat{w}_2^{(f)}, \ldots, \hat{w}_m^{(f)})$ is a solution.

Second, for $f_1, f_2 \in E_t$, if $f_1 \subseteq f_2$, then it is obvious that $\hat{\mathbf{w}}^{(f_1)} \geq \hat{\mathbf{w}}^{(f_2)}$ from the definition of $\hat{\mathbf{w}}^{(f)}$.

Third, for any $\mathbf{r} \in \zeta(A, \mathbf{b})$, we shall show that there exists a $f \in F$ such that $\mathbf{r} \leq \hat{\mathbf{w}}^{(f)}$. Let $N'(i) = \{j \in N_m : a_{ij}\theta r_j = b_i\}$, then $N'(i) \neq \emptyset$ and $N'(i) \subseteq N(i)$. Construct $E_r = \{f \subseteq N_m : \forall i \in N_n, f \cap N'(i)$ contains at least one element$\}$, then it is obvious that $E_r \subseteq E_t$. We show that $\mathbf{r} \leq \hat{\mathbf{w}}^{(f)}, \forall f \in E_r$, then there exists a $f \in F$ such that $\mathbf{r} \leq \hat{\mathbf{w}}^{(f)}$. For any $f \in E_r$, if $\forall j \in f$, then $\forall j \in N'(i), a_{ij}\theta r_j = b_i$. Then $r_j \leq a_{ij}\hat{\otimes}^{\theta} b_i$, hence, $r_j \leq \inf\{a_{ij}\hat{\otimes}^{\theta} b_i : j \in N(i)\} = \hat{w}_j^{(f)}$. Therefore, $\mathbf{r} \leq \hat{\mathbf{w}}^{(f)}, \forall f \in E_r$.

Finally, we show that $M = \{\hat{\mathbf{w}}^{(f)} : \forall f \in F\}$ is the set of maximal solutions of the FRE of the form (2). The left is only show that the elements of M are not comparable with each other. Since F is minimal, for any $f_1, f_2 \in F$, there are $j_1 \in f_1$ and $j_2 \in f_2$ such that $j_1 \notin f_2, j_2 \notin f_1$. In this case, the j_1th coordinate of $\hat{\mathbf{w}}^{(f_1)}$ is 1 and the j_2th coordinate of $\hat{\mathbf{w}}^{(f_2)}$ is 1, while the j_2th coordinate of $\hat{\mathbf{w}}^{(f_1)}$ and the j_1th coordinate of $\hat{\mathbf{w}}^{(f_2)}$ is not equal to 1. This implies that $\hat{\mathbf{w}}^{(f_1)}$ and $\hat{\mathbf{w}}^{(f_2)}$ are not comparable with each other. Thus, M is the maximal subset of $\zeta(A, \mathbf{b})$. □

4 Simulation Results

Example. The training set $D = \{(\mathbf{a}_j, b_j), j = 1, 2, 3, 4\}$ is as follows:

$$\begin{matrix} [0.7 & 0.8 & 0.7] & 0.3 \\ [0.4 & 0.7 & 0.6] & 0.5 \\ [0.3 & 0.1 & 0.3] & 0.7 \\ [0.2 & 0.1 & 0.1] & 0.8 \end{matrix}$$

Let θ be a Kleene-Dienes implication, i.e. $a\theta b = (1-a) \vee b, \forall a, b \in [0, 1]$.

Let the error constant $\epsilon = 0$ and the learning rate $\alpha = 1$, $\alpha_1 = 1, \alpha_2 = 0$. First initialize the weights $\breve{w}_j^{(1)}(0) = 0, j = 1, 2, 3, 4$. From the Algorithm 1, we

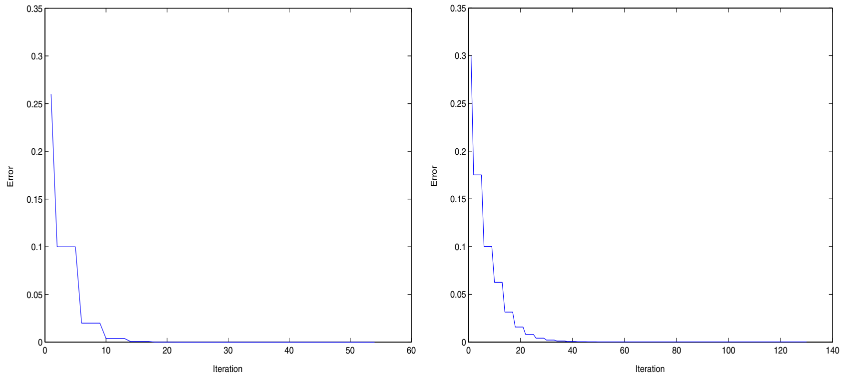

Fig. 3. The result of Algorithm 1 for initial conditions $\alpha = 0.8$(left) and $\alpha = 0.5$(right)

Table 1. The relation between α and iterations ($\epsilon = 1.0e - 10$)

α	Iterations
0.9	38
0.8	54
0.7	74
0.6	94
0.5	126

can get the following weight vector after 2 iterations:

$$\breve{\mathbf{w}} = (0 \;\; 0.5 \;\; 0.5)^T,$$

that is, the least solution of the equation $A \circ^\theta R = \mathbf{b}$, where

$$A = \begin{pmatrix} 0.7 & 0.8 & 0.7 \\ 0.4 & 0.7 & 0.6 \\ 0.3 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.1 \end{pmatrix}, \mathbf{b} = (0.3 \;\; 0.5 \;\; 0.7 \;\; 0.8)^T.$$

In the following, we calculate all maximal solutions of the above equation using Algorithm 2. First initialize the weights $\hat{w}_j^{(1)}(0) = 1, j = 1, 2, 3, 4$. Let the error constant $\epsilon = 0$ and the learning rate $\eta = 1$. After 4 iterations, we can get the adjusted weight vector is $\hat{\mathbf{w}}^{(4)} = (0.3 \;\; 0.5 \;\; 0.5)^T$, and $N(4) = \{1\}$. In this case, $E_t = \{\{1, 2\}, \{1, 2, 3\}, \{1, 3\}\}$, and then $F = \{\{1, 2\}, \{1, 3\}\}$. Thus, the adjusted weight vector is $\hat{\mathbf{w}}^{(f_1)} = (0.3 \;\; 0.5 \;\; 1)^T, \hat{\mathbf{w}}^{(f_2)} = (0.3 \;\; 1 \;\; 0.5)^T$, where $f_1 = \{1, 2\}, f_2 = \{1, 3\}$. Thus, the maximal solution set $M = \{(0.3 \;\; 0.5 \;\; 1)^T, (0.3 \;\; 1 \;\; 0.5)^T\}$.

Let $\alpha = 0.8(0.5)$ and using Algorithm 1 with $\epsilon = 1.0e - 10$ after 54 (126) iterations it converges to:

$$\breve{\mathbf{w}} = (0 \;\; 0.5 \;\; 0.5)^T,$$

the change of the error is illustrated in Fig. 3.

These simulation results also illustrate that the last stable points of the network are good approximate solutions when the FRE has no solution.

5 Conclusions

In the present paper, we have studied the resolution problem of min-implication FREs based on neural network. Due to the relatively complement of neural networks and fuzzy inference systems, the adaptation ability of system paraments in fuzzy inference systems is greatly improved. Both the least solution and all maximial solutions are obtained by taking the proposed learning algorithms when the solution set of the FRE is nonempty, otherwise, the learning of the neural network converges to good approximate solutions. Finally, we demonstrate them in detail, and provide some simulation results to get the last stable points. The results of the present paper can be applied into fuzzy modeling.

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